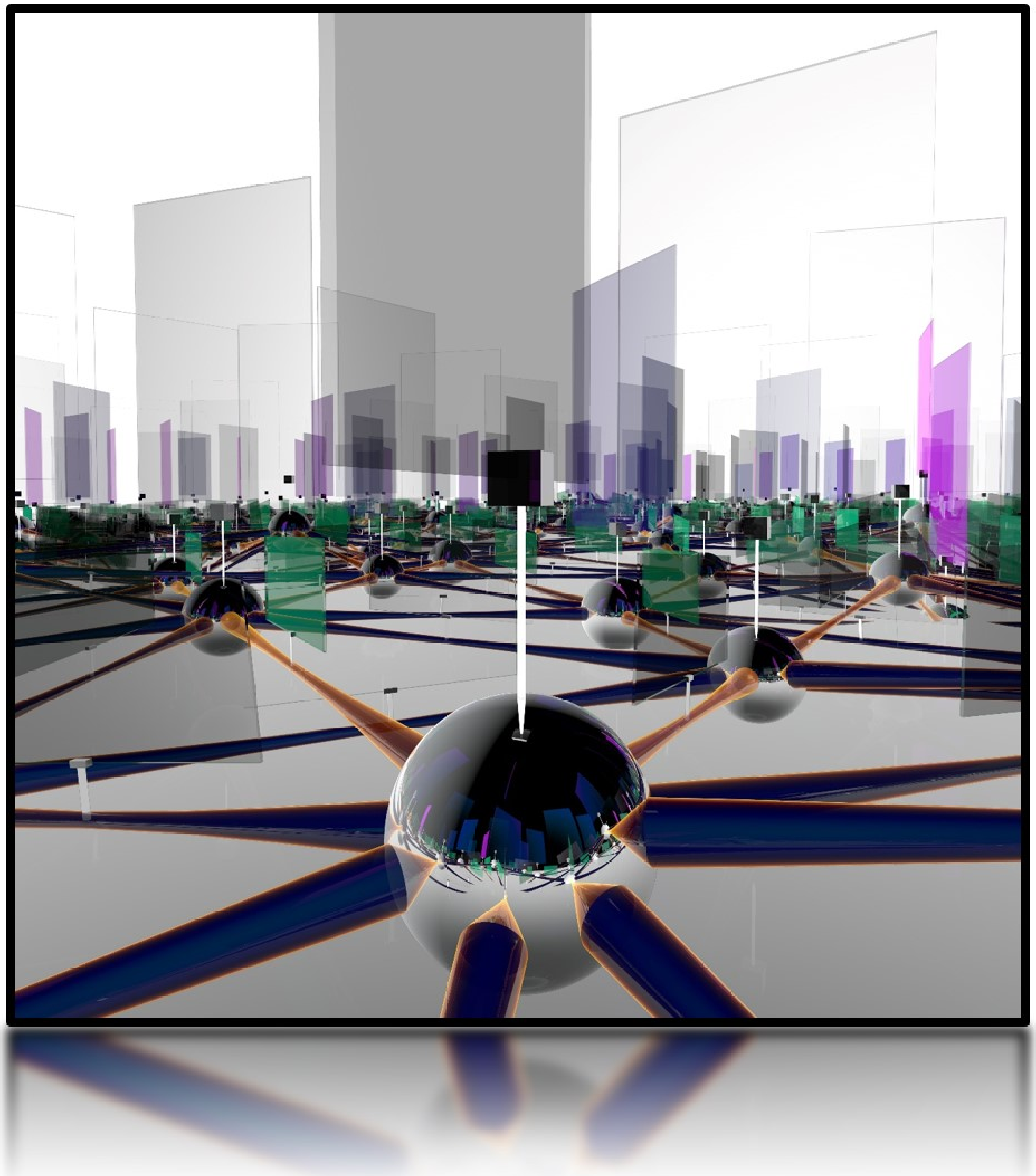
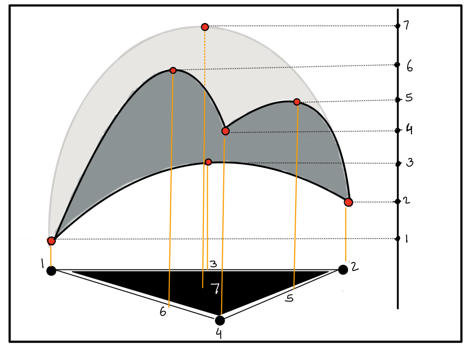
7. SHEAVES



7.1 Fibers and Persistence



Let $f : K \to \mathbb{R}_+$ be a function that assigns a non-negative real number $f(\sigma)$ to every simplex σ of a simplicial complex K. We call f *monotone* if it satisfies $f(\sigma) \leq f(\tau)$ whenever σ is a face of τ in K. Specifying a monotone f is equivalent to imposing an $\mathbb{R}_+$-indexed filtration $\boldsymbol{F}_\bullet$ on K — to discover this filtration, one uses the rule $\boldsymbol{F}_t K = \{\sigma \in K \mid f(\sigma) \leq t\}$. We call $\boldsymbol{F}$ the **sublevelset filtration** of K with respect to f. Conversely, if we are given a filtration $\boldsymbol{F}_\bullet$ of K, then the corresponding monotone function $f : K \to \mathbb{R}_+$ is given by $f(\sigma) = \inf\{t \in \mathbb{R}_+ \mid \sigma \in \boldsymbol{F}_t K\}$. Thus, much of persistent homology (particularly its application to the study of filtered simplicial complexes) can be interpreted as the systematic analysis of homology groups associated to certain *fibers* of f — for each $t \in \mathbb{R}^+$, the fiber of interest is a subcomplex of K:

$$\{f \leq t\} := \{\sigma \in K \mid 0 \leq f(\sigma) \leq t\}$$

Thanks to the finiteness of K, taking the k-th homology of sublevelset filtrations always produces tame persistence modules (in the sense of Definition 6.13); thus these modules admit a barcode decomposition as guaranteed by Corollary 6.14. These barcodes satisfy two special properties: first, they allow us to combinatorially describe the homology of each fiber $\{f \leq t\}$ and the rank of the linear maps

$$\mathbf{H}_k\big(\{f \leq t\}\big) \to \mathbf{H}_k\big(\{f \leq s\}\big)$$

induced on k-th homology by inclusion of fibers for all pairs $t \leq s$. Second, if we have a another monotone function $f' : K \to \mathbb{R}$ that is ϵ-close to our f, i.e., if we have

$$|f(\sigma) - f'(\sigma)| < \epsilon \text{ for every } \sigma \text{ in } K,$$

then the barcodes for f' will be no more than ϵ-apart from those of f with respect to the bottleneck distance (see Definition 6.18 and Exercise 7.1). Thus, all intervals longer than 2ϵ in the barcode of f correspond to fiber homology classes that are stable with respect to ϵ-perturbations of f.

Card-carrying mathematicians will immediately wonder whether similar stability results can be obtained for maps $K \to X$ when X is more complicated than $\mathbb{R}_+$: *ars gratia artis*. Those with the ability to withstand this temptation to generalize might instead be compelled by more practical considerations. A monotone map $f : K \to \mathbb{R}_+$ associates a (real-valued) measurement to each simplex, and we are often interested in several such measurements $\{f_i : K \to \mathbb{R}_+ \mid 1 \leq i \leq n\}$ and wish to study (the homology of) their common sublevelsets $\bigcap_{i=1}^n \{f_i \leq t_i\}$ simultaneously. Thus, we may as well assign

$$\sigma \mapsto \big(f_1(\sigma), \ldots, f_n(\sigma)\big)$$

and study the fibers of this single vector-valued map $K \to \mathbb{R}^n_+$.

Even more interesting from a topological viewpoint is the scenario where the f_i associate angles in $[0, 2\pi)$ to simplices; in this case, we have a map $f : K \to \mathbb{T}^n$ to the n-torus (i.e., the product of n circles). Now it no longer makes sense to seek monotonicity or ask about fibers of the form $\{f_i \leq t_i\}$, since there is no natural partial order on points of the n-torus. On the other hand, we can certainly triangulate the torus so that f becomes a simplicial map and study the fiberwise homology of f over simplices (or subcomplexes) of $\mathbb{T}^n$. It is, therefore, in our interest to understand the (co)homology groups of fibers of simplicial maps $f : K \to L$. The optimal data structure which coherently organizes these fiber homology groups is called a **sheaf**.

7.2 SHEAVES

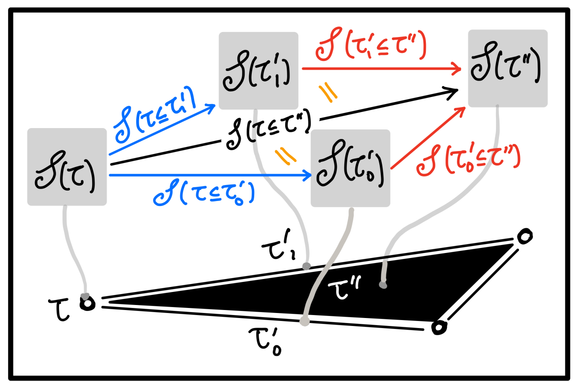
Let L be a simplicial complex and $\mathbb{F}$ a field. We write $(L, \leq)$ to denote the poset of simplices in L ordered by the face relation.

DEFINITION 7.1. A **sheaf** over L is a functor $\mathscr{S} : (L, \leq) \to \mathbf{Vect}_{\mathbb{F}}$. In other words, $\mathscr{S}$ assigns

(1) to each simplex τ of L an $\mathbb{F}$-vector space $\mathscr{S}(\tau)$ called the *stalk*, and
(2) to each $\tau \leq \tau'$ in L a linear map $\mathscr{S}(\tau \leq \tau') : \mathscr{S}(\tau) \to \mathscr{S}(\tau')$ called the *restriction map*,

subject to the usual (identity and associativity) categorical axioms:

(1) for every simplex τ in L, the map $\mathscr{S}(\tau \leq \tau)$ is the identity on $\mathscr{S}(\tau)$, and
(2) for every triple $\tau \leq \tau' \leq \tau''$ in L, we have $\mathscr{S}(\tau' \leq \tau'') \circ \mathscr{S}(\tau \leq \tau') = \mathscr{S}(\tau \leq \tau'')$.



We call L the **base space** of the sheaf $\mathscr{S}$. From a purely algebraic perspective, $\mathscr{S}$ is an arrangement of $\mathbb{F}$-vector spaces and linear maps parametrized by the simplices of L and their face relations. Alternately, one may view $\mathscr{S}$ as a gadget which weights these simplices and face relations by vector spaces and linear maps respectively. Although the stalks of a sheaf can vary drastically from simplex to simplex, the associativity requirement places severe constraints on restriction maps. For instance, both composite paths from $\mathscr{S}(\tau)$ to $\mathscr{S}(\tau'')$ in the accompanying figure must evaluate to $\mathscr{S}(\tau \leq \tau'')$. On the other hand, if L is one-dimensional then associativity holds automatically because there are no ascending triples $\tau < \tau' < \tau''$ of simplices.

EXAMPLE 7.2. Here are three examples of sheaves on a simplicial complex L, in increasing order of complexity.

(1) The **zero** sheaf $\underline{0}_L$, as suggested by its name, assigns the trivial (i.e., zero-dimensional) $\mathbb{F}$-vector space to every simplex. This forces all the restriction maps to also be zero.
(2) Given a simplex τ of L, the associated **skyscraper** sheaf $\underline{\mathrm{Sk}}_\tau$ over L assigns the trivial vector space to every simplex except τ, whose stalk is the one-dimensional vector space $\mathbb{F}$. The restriction map associated to $\tau \leq \tau$ is the identity, while all other restriction maps must be zero.
(3) The **constant** sheaf $\underline{\mathbb{F}}_L$ assigns the one-dimensional stalk $\mathbb{F}$ to every simplex of L and the identity restriction map $\mathbb{F} \to \mathbb{F}$ to every face relation in sight.

More interesting examples will become available later.

As mentioned in the previous Section, our main interest in sheaves comes from their remarkable ability to encode the homology groups of fibers of simplicial maps. Recall from (2) that the fiber of a simplicial map $f : K \to L$ under a simplex τ of L is the subcomplex of K given by

$$\tau/f = \{\sigma \in K \mid f(\sigma) \leq \tau\}.$$

And moreover, for any pair $\tau \leq \tau'$ in L there is an obvious inclusion of fibers $\tau/f \hookrightarrow \tau'/f$ because any σ in K satisfying $f(\sigma) \leq \tau$ automatically satisfies $f(\sigma) \leq \tau'$. Thus, fitting the homology groups $\mathbf{H}_k(\tau/f; \mathbb{F})$ into a sheaf over L becomes a matter of invoking the functoriality of homology with respect to inclusion maps.

PROPOSITION 7.3. *Let $f : K \to L$ be a simplicial map. For each dimension $k \geq 0$, the assignments*

$$\tau \mapsto \mathbf{H}_k(\tau/f), \text{ and}$$
$$(\tau \leq \tau') \mapsto \mathbf{H}_k(\tau/f \hookrightarrow \tau'/f)$$

constitute a sheaf over L, which we denote $\mathscr{F}_f^k$ and call the k-th **fiber homology sheaf** *of f.*

The proof is not complicated — for any triple of simplices $\tau \leq \tau' \leq \tau''$ in L, the inclusion $\tau/f \hookrightarrow \tau''/f$ factors through τ'/f; the identity and associativity axioms of Definition 7.1 are satisfied simply because homology is functorial. It should also be noted that in general some fiber τ/f might be empty, in which case we would have $\mathscr{F}_f^k(\tau) = \mathbf{H}_k(\tau/f) = 0$ for all k.

EXAMPLE 7.4. Fiber homology sheaves of the identity simplicial map $\text{id} : L \to L$ are already familiar to us — for each simplex τ of L, the fiber τ/id is the subcomplex $\overline{\tau}$ consisting of the single simplex τ along with all of its faces. Each such fiber is contractible by Proposition 2.6, and hence has the homology of a point $\Delta(0)$. Consequently,

$$\mathscr{F}_{\text{id}}^k(\tau) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Thus, $\mathscr{F}_{\text{id}}^k$ is the zero sheaf $\underline{0}_L$ whenever $k > 0$. With a bit of effort, one can discover that the restriction maps of $\mathscr{F}_{\text{id}}^0$ are all identities $\mathbb{F} \to \mathbb{F}$, and so $\mathscr{F}_{\text{id}}^0$ is the constant sheaf $\underline{\mathbb{F}}_L$.

Those experiencing nostalgia for persistent homology have no cause for concern: every sheaf $\mathscr{S}$ is filled to the brim with persistence modules. Take any ascending sequence

$$\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n$$

of simplices in the base space L, and note that the restriction maps produce a persistence module

$$\mathscr{S}(\tau_0) \xrightarrow{\mathscr{S}(\tau_0 \leq \tau_1)} \mathscr{S}(\tau_1) \xrightarrow{\mathscr{S}(\tau_1 \leq \tau_2)} \cdots \xrightarrow{\mathscr{S}(\tau_{n-1} \leq \tau_n)} \mathscr{S}(\tau_n).$$

It follows from the associativity axiom of Definition 7.1 that the number of intervals $[i, j]$ in the barcode of this persistence module must equal the rank of $\mathscr{S}(\tau_i \leq \tau_j)$.

7.3 SHEAF COHOMOLOGY

Taking the perspective of sheaves as *algebraic weights on simplices* seriously produces a suite of new cohomology theories for simplicial complexes. To define these sheaf-infused cohomology groups, we must first build a suitable cochain complex using the data of a sheaf; to this end, fix a sheaf $\mathscr{S}$ on a simplicial complex L.

DEFINITION 7.5. For each dimension $k \geq 0$, the vector space of **k-cochains of L with $\mathscr{S}$-coefficients** is the product

$$\mathbf{C}^k(L; \mathscr{S}) = \prod_{\dim \tau = k} \mathscr{S}(\tau)$$

of the stalks of $\mathscr{S}$ over all the k-dimensional simplices of L.

Depending on which sheaf $\mathscr{S}$ is being used as the *coefficient system* in the definition above, the cochain groups $\mathbf{C}^\bullet(L; \mathscr{S})$ might be quite different from the familiar simplicial cochain groups $\mathbf{C}^\bullet(L; \mathbb{F})$ of Definition 5.1 — for instance, when $\mathscr{S} = \underline{0}_L$, we obtain trivial cochain groups in all dimensions regardless of L. But for $\mathscr{S} = \underline{\mathbb{F}}_L$, we recover the usual simplicial cochain groups of

L. The key point is that while the constant sheaf identifies a unique one-dimensional subspace of $\mathbf{C}^k(L)$ with every k-simplex of L, using a different sheaf $\mathscr{S}$ allows us to upgrade the contribution of some simplices (by assigning them stalks of dimension > 1) and diminishing the contribution of others (by assigning them zero stalks).

Let's assume that the vertices of L are ordered so that each k-simplex τ has a well-defined i-th face τ_{-i} for i in $\{0, \ldots, k\}$ (see Definition 3.4). For each pair of simplices τ, τ' in L we write

$$[\tau : \tau'] := \begin{cases} +1 & \tau = \tau'_{-i} \text{ for } i \text{ even}, \\ -1 & \tau = \tau'_{-i} \text{ for } i \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $[\tau : \tau'] \in \mathbb{F}$ is precisely the coefficient of τ' in the simplicial coboundary of τ, or equivalently, the coefficient of τ in the simplicial boundary of τ'.

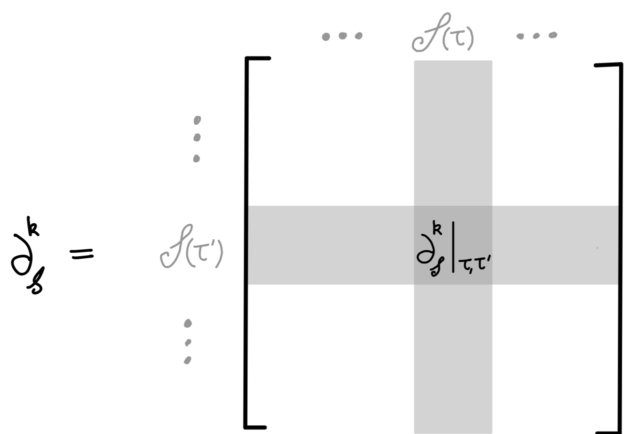
DEFINITION 7.6. For each $k \geq 0$, the k-th **coboundary map** of L **with $\mathscr{S}$-coefficients** is the linear map

$$\partial^k_{\mathscr{S}} : \mathbf{C}^k(L; \mathscr{S}) \to \mathbf{C}^{k+1}(L; \mathscr{S})$$

defined via the following block-action: for each pair of simplices $\tau \leq \tau'$ with $\dim \tau = k$ and $\dim \tau' = k + 1$, the $\mathscr{S}(\tau) \to \mathscr{S}(\tau')$ component of $\partial^k_{\mathscr{S}}$ is given by

$$\partial^k_{\mathscr{S}}|_{\tau,\tau'} = [\tau : \tau'] \cdot \mathscr{S}(\tau \leq \tau') \tag{6}$$

From a computational perspective, it often helps to view $\partial^k_{\mathscr{S}}$ as an enormous block-matrix whose columns are indexed by (stalks of) all the k-simplices in L and rows are indexed by (stalks of) all the $(k+1)$-simplices; the component $\partial^k_{\mathscr{S}}|_{\tau,\tau'}$ is the block in the column of τ and the row of τ':



The expression (6) for $\partial^k_{\mathscr{S}}|_{\tau,\tau'}$ involves a restriction map, but note that it makes sense even when τ is not a face of τ': in this case, the scalar $[\tau : \tau']$ is zero, so the entire block is zero.

REMARK 7.7. If $\mathscr{S}$ is the constant sheaf $\underline{\mathbb{F}}_L$, then all the rows and columns have width one (since all the stalks are one-dimensional); and since the restriction maps in this case are all identities, the entry $\partial^k_{\mathscr{S}}|_{\tau,\tau'}$ lies in $\{0, \pm 1\}$ depending on whether or not τ is a face of τ'. Thus, both $\mathbf{C}^k(L; \mathscr{S})$ and $\partial^k_{\mathscr{S}}$ reduce to the familiar objects from Definition 5.1 when $\mathscr{S} = \underline{\mathbb{F}}_L$.

The harsh constraints placed on restriction maps of $\mathscr{S}$ by the associativity axiom of Definition 7.1 will now start yielding rich dividends. The following result establishes that the choice of terminology (cochains and coboundary operators) for the objects $\mathbf{C}^k(L; \mathscr{S})$ and $\partial^k_{\mathscr{S}}$ is apposite.

PROPOSITION 7.8. *The sequence*

$$0 \longrightarrow \mathbf{C}^0(L;\mathscr{S}) \xrightarrow{\partial^0_{\mathscr{S}}} \mathbf{C}^1(L;\mathscr{S}) \xrightarrow{\partial^1_{\mathscr{S}}} \cdots \xrightarrow{\partial^{k-1}_{\mathscr{S}}} \mathbf{C}^k(L;\mathscr{S}) \xrightarrow{\partial^k_{\mathscr{S}}} \mathbf{C}^{k+1}(L;\mathscr{S}) \xrightarrow{\partial^{k+1}_{\mathscr{S}}} \cdots$$

forms a cochain complex over $\mathbb{F}$. In other words, $\partial^k_{\mathscr{S}} \circ \partial^{k-1}_{\mathscr{S}}$ equals zero for all $k \geq 1$.

PROOF. It suffices to verify that the composite of two adjacent coboundary operators equals zero block-wise. Namely, for each $(k-1)$-simplex τ and $(k+1)$-simplex τ'' we will show that the $\mathscr{F}(\tau) \to \mathscr{F}(\tau'')$ block of this composite is the zero map, from which the desired conclusion immediately follows. For any vector v in $\mathscr{F}(\tau)$, we calculate

$$\begin{aligned}
\partial^k_{\mathscr{S}} \circ \partial^{k-1}_{\mathscr{S}}(v) &= \sum_{\dim \tau' = k} \partial^k_{\mathscr{S}}|_{\tau',\tau''} \circ \partial^{k-1}_{\mathscr{S}}|_{\tau,\tau'}(v) && \text{by Definition 7.6} \\
&= \sum_{\tau<\tau'<\tau''} \partial^k_{\mathscr{S}}|_{\tau',\tau''} \circ \partial^{k-1}_{\mathscr{S}}|_{\tau,\tau'}(v) && \text{eliminating zero terms} \\
&= \sum_{\tau<\tau'<\tau''} [\tau' : \tau''] \cdot [\tau : \tau'] \cdot \mathscr{S}(\tau' \leq \tau'') \circ \mathscr{S}(\tau \leq \tau')(v) && \text{by (6)} \\
&= \sum_{\tau<\tau'<\tau''} [\tau' : \tau''] \cdot [\tau : \tau'] \cdot \mathscr{S}(\tau \leq \tau'')(v) && \text{associativity axiom!} \\
&= \left(\sum_{\tau<\tau'<\tau''} [\tau' : \tau''] \cdot [\tau : \tau'] \right) \cdot \mathscr{S}(\tau \leq \tau'')(v) && \text{collecting scalars}
\end{aligned}$$

But now the scalar in parentheses is zero because it equals the coefficient of τ'' in the composite $\partial^k_L \circ \partial^{k-1}_L(\tau)$. Since our choice of v was arbitrary, the composite $\partial^k_{\mathscr{S}} \circ \partial^{k-1}_{\mathscr{S}}$ is identically zero as desired. ☐

Having produced a cochain complex from $\mathscr{S}$, we can safely define the associated cohomology groups in the usual fashion.

DEFINITION 7.9. For each dimension $k \geq 0$, the k-th **cohomology group of L with coefficients in $\mathscr{S}$** is the quotient vector space

$$\mathbf{H}^k(L;\mathscr{S}) = \ker \partial^k_{\mathscr{S}} / \operatorname{img} \partial^{k-1}_{\mathscr{S}}.$$

At the moment, this definition is simply a way of producing cohomology groups from sheaves. We know, based on the discussion above, that this *sheaf cohomology* agrees with standard cohomology whenever $\mathscr{S}$ is the constant sheaf $\underline{\mathbb{F}}_L$. It is challenging to visualize sheaf cohomology for more general choices of $\mathscr{S}$; but in the next Section, we will provide a topological interpretation for the simplest sheaf cohomology group $\mathbf{H}^0(L;\mathscr{S})$ for arbitrary $\mathscr{S}$.

7.4 THE ÉTALE SPACE AND SECTIONS

Let L be a simplicial complex and $\mathscr{S}$ a sheaf on L; both will remain fixed throughout this section. We recall that the geometric realization of every simplex τ in L is denoted $|\tau| \subset |L|$ (see Definition 1.7) and its open star (from Definition 1.17) is denoted $\mathbf{st}(\tau) \subset L$. The realization of this open star is

$$|\,\mathbf{st}(\tau)| = \bigcup_{\tau \leq \tau'} |\tau'|^\circ,$$

where $|\tau'|^\circ$ stands for the interior of $|\tau'|$ in $|L|$. For each $x \in |L|$ there is a unique simplex $\tau \in L$ with $x \in |\tau|^\circ$, which we will denote by τ_x throughout this section.

DEFINITION 7.10. The **étale space** of a sheaf $\mathscr{S}$ on L is the topological space $\mathbf{E}\mathscr{S}$ defined as follows. Its underlying set consists of pairs

$$\mathbf{E}\mathscr{S} = \{(x,v) \mid v \in \mathscr{S}(\tau_x)\}.$$

A basis for the topology is prescribed by open sets $U_{\tau,v} \subset \mathbf{E}\mathscr{S}$ indexed by pairs (τ, v) where $\tau \in L$ is a simplex and $v \in \mathscr{S}(\tau)$ is a vector lying in its stalk. Each such *basic open* set is:

$$U_{\tau,v} = \{(x,w) \mid \tau_x \geq \tau \text{ and } w = \mathscr{S}(\tau \leq \tau_x)(v)\}.$$

There is a natural projection $\pi_{\mathscr{S}} : \mathbf{E}\mathscr{S} \twoheadrightarrow |L|$ sending each (x, v) to x; this is called the **étale map** of $\mathscr{S}$ and it satisfies two strong properties. First, its restriction to each basic open $U_{\tau,v}$ is a homeomorphism onto $|\,\mathbf{st}(\tau)|$. And second, for each x in L we have

$$\pi_{\mathscr{S}}^{-1}(x) = \{x\} \times \mathscr{S}(\tau_x).$$

Thus, $\pi_{\mathscr{S}}^{-1}(x)$ has the structure of a vector space for each x in $|L|$. The étale space is is home to some very special subspaces; these can be discovered by attempting to find right-inverses for the affiliated étale map.

DEFINITION 7.11. Let $L' \subset L$ be any subcollection of simplices (which do not necessarily form a subcomplex). A **section of $\mathscr{S}$ over** L' is any continuous map $h : |L'| \to \mathbf{E}\mathscr{S}$ for which the composite $\pi_{\mathscr{S}} \circ h$ equals the identity map on $|L'|$. The set of all such sections is denoted $\Gamma(L'; \mathscr{S})$.

The case $L = L'$ is of particular interest — we call $\Gamma(L; \mathscr{S})$ the set of **global sections** of $\mathscr{S}$. Since any section h in $\Gamma(L', \mathscr{S})$ satisfies $\pi_{\mathscr{S}} \circ h = \text{id}$, it must at least send each point x of $|L'|$ to a vector $h(x)$ in the stalk $\mathscr{S}(\tau_x)$. Since h is also continuous, we can make two stronger claims.

PROPOSITION 7.12. *For any subcollection $L' \subset L$ of simplices,*

(1) *each section h in $\Gamma(L'; \mathscr{S})$ is constant on $|\tau|^\circ$ for each τ in L'; moreover,*
(2) *the set $\Gamma(L'; \mathscr{S})$ has the structure of a vector space.*

PROOF. Fix any simplex τ in L'. Since $\pi_{\mathscr{S}} \circ h$ is the identity, it follows that $h(|\,\mathbf{st}(\tau)|)$ is a subset of $\pi_{\mathscr{S}}^{-1}(|\,\mathbf{st}(\tau)|)$. By definition, there is a decomposition

$$\pi_{\mathscr{S}}^{-1}(|\,\mathbf{st}(\tau)|) \simeq \coprod_{v \in \mathscr{S}(\tau)} U_{\tau,v},$$

where each $U_{\tau,v}$ is a basic open set. Since h is continuous and $|\,\mathbf{st}(\tau)|$ is connected, there is a single v in $\mathscr{S}(\tau)$ so that $h(|\,\mathbf{st}(\tau)|) \subset U_{\tau,v}$. Thus, any two points x and x' in $|\tau|^\circ$ are sent by h to the same vector $\mathscr{S}(\tau \leq \tau)(v) = v$, which proves the first claim. Armed with this knowledge, we may as well view h as a function sending each simplex $\tau \in L'$ to a vector $h(\tau) \in \mathscr{S}(\tau)$. With this shift in perspective, the vector space structure on $\Gamma(L'; \mathscr{S})$ becomes obvious: for any pair of scalars α, β in $\mathbb{F}$ and sections h, g in $\Gamma(L'; \mathscr{S})$, we can form the linear combination $\alpha \cdot h + \beta \cdot g$ that sends each τ to the vector $\alpha \cdot h(\tau) + \beta \cdot g(\tau)$ in $\mathscr{S}(\tau)$. □

Writing sections as assignments of stalk-vectors to simplices of L' (rather than to points of $|L'|$) allows us to view them as finite objects. Implicit in the proof of the above result is the following observation, which establishes that sections correspond to choices of stalk-vectors that are **compatible** with respect to the restriction maps of $\mathscr{S}$.

COROLLARY 7.13. *If h is a section in $\Gamma(L', \mathscr{S})$, then for every pair of simplices $\tau \leq \tau'$ in L' we have the equality*

$$\mathscr{S}(\tau \leq \tau')(h(\tau)) = h(\tau').$$

We have been discussing sections of sheaves because they are intimately related to the sheaf cohomology groups from Definition 7.9.

THEOREM 7.14. *For any sheaf $\mathscr{S}$ over a simplicial complex L, there is a vector space isomorphism*

$$\mathbf{H}^0(L; \mathscr{S}) \simeq \Gamma(L; \mathscr{S})$$

between the zeroth cohomology groups of L with coefficients in $\mathscr{S}$ and the global sections of $\mathscr{S}$.

PROOF. Although this proof has been assigned as an exercise, we show the first step of the argument here as a (substantial) hint. The zeroth cohomology $\mathbf{H}^0(L; \mathscr{S})$ is precisely the kernel of the coboundary map $\partial^0_{\mathscr{S}}$, whose block structure has been described in Definition 7.6. The row-blocks are indexed by the 1-simplices, each of which contains exactly two vertices in its boundary. The row corresponding to a 1-simplex $\tau = (u_0, u_1)$ can only have nonzero blocks in the two columns corresponding to its vertices u_0 and u_1. Thus, a cochain v in $\mathbf{C}^0(L; \mathscr{S})$ lies in the kernel of this coboundary matrix if and only if its components $v_i \in \mathscr{S}(u_i)$ for i in $\{0, 1\}$ satisfy

$$\mathscr{S}(u_0 \leq \tau)(v_0) = \mathscr{S}(u_1 \leq \tau)(v_1).$$

This is the first step in showing that v constitutes a section. □

REMARK 7.15. When defining sections of $\mathscr{S}$ over subsets of L, we only used the topology of $\mathbf{E}\mathscr{S}$ and properties of the map $\pi_{\mathscr{S}} : \mathbf{E}\mathscr{S} \twoheadrightarrow |L|$. In fact, one can completely recover $\mathscr{S}$ from its étale map: the stalk $\mathscr{S}(\tau)$ over each simplex τ of L is the vector space of sections $\Gamma(|\,\mathbf{st}(\tau)|; \mathscr{S})$ over its open star, and the restriction map associated to $\tau \leq \tau'$ is obtained by using the fact that every section $|\,\mathbf{st}(\tau)| \to \mathbf{E}\mathscr{S}$ restricts to a section over the smaller set $|\,\mathbf{st}(\tau')|$.

7.5 PUSHFORWARDS AND PULLBACKS

There is a natural way to define maps of sheaves over a fixed simplicial complex L.

DEFINITION 7.16. A **morphism of sheaves** $\Phi_\bullet : \mathscr{S} \to \mathscr{S}'$ over L consists of linear maps $\Phi_\tau : \mathscr{S}(\tau) \to \mathscr{S}'(\tau)$ indexed by simplices $\tau \in L$ so that the following diagram of vector spaces commutes for each $\tau \leq \tau'$:

$$\begin{array}{ccc} \mathscr{S}(\tau) & \xrightarrow{\Phi_\tau} & \mathscr{S}'(\tau) \\ {\scriptstyle \mathscr{S}(\tau \leq \tau')}\downarrow & & \downarrow{\scriptstyle \mathscr{S}'(\tau \leq \tau')} \\ \mathscr{S}(\tau') & \xrightarrow[\Phi_{\tau'}]{} & \mathscr{S}'(\tau'). \end{array}$$

These morphisms endow the set of all sheaves over L with the structure of a category, which we will denote by $\mathbf{Sh}(L)$. Sheaf morphisms induce well-defined maps on sheaf cohomology (this is an exercise to this Chapter).

Our goal here is to show how sheaves can be transported back and forth between a pair of simplicial complexes K and L by using a simplicial map $f : K \to L$. Surprisingly, the easier direction is backwards: we can construct a sheaf on K from a sheaf on L in a relatively straightforward manner.

DEFINITION 7.17. The **pullback** of a sheaf $\mathscr{S}$ over L across the simplicial map $f : K \to L$ is a new sheaf $f^*\mathscr{S}$ over K defined as follows. The stalk over every simplex σ in K is

$$f^*\mathscr{S}(\sigma) = \mathscr{S}(f(\sigma)),$$

while the restriction map for $\sigma \leq \sigma'$ is

$$f^*\mathscr{S}(\sigma \leq \sigma') = \mathscr{S}(f(\sigma) \leq f(\sigma'))$$

Transporting sheaves from K forwards to L along $f : K \to L$ is more intricate, because now the direction of f works against us. For each simplex τ of L, there might be a large collection of simplices in K which get mapped to (a co-face of) τ; we must somehow combine the $\mathscr{T}$-stalks over all these simplices in order to produce a sheaf over K. Here it helps to utilize the perspective from Remark 7.15 and define the desired sheaf in terms of its étale space.

DEFINITION 7.18. The **pushforward** of a sheaf $\mathscr{T}$ on K along a simplicial map $f : K \to L$ is a new sheaf $f_*\mathscr{T}$ on L whose étale space equals

$$\mathbf{E}f_*\mathscr{T} = \left\{ \left(|f|(x), v\right) \mid (x, v) \in \mathbf{E}\mathscr{T} \right\};$$

here $|f| : |K| \to |L|$ is the continuous map induced by f.

By our recipe for extracting sheaves from their étale spaces, it follows that the stalk $f_*\mathscr{T}(\tau)$ for each simplex τ of L is the vector space of sections $\Gamma(|f/\tau|; \mathscr{T})$, where f/τ is the *dual* fiber

$$f/\tau = \{\sigma \in K \mid f(\sigma) \geq \tau\}.$$

Although this dual fiber is not generally a subcomplex of K unlike τ/f, the space of $\mathscr{T}$'s sections over it is still well-defined.

REMARK 7.19. Pullbacks and pushforwards are functors between $\mathbf{Sh}(K)$ and $\mathbf{Sh}(L)$ — so, we can pull and push not only sheaves but also their morphisms. Moreover, they form a dual *adjoint pair* in the following sense. Given a simplicial map $f : K \to L$ along with sheaves $\mathscr{S} \in \mathbf{Sh}(L)$ and $\mathscr{T} \in \mathbf{Sh}(K)$, there is a bijection

$$\begin{bmatrix} \text{Morphisms} \\ f^*\mathscr{S} \to \mathscr{T} \\ \text{in } \mathbf{Sh}(K) \end{bmatrix} \simeq \begin{bmatrix} \text{Morphisms} \\ \mathscr{S} \to f_*\mathscr{T} \\ \text{in } \mathbf{Sh}(L) \end{bmatrix}$$

To prove this, one must first discover natural sheaf morphisms

$$\mathscr{S} \to f_*f^*\mathscr{S} \quad \text{and} \quad f^*f_*\mathscr{T} \to \mathscr{T}$$

in $\mathbf{Sh}(L)$ and $\mathbf{Sh}(K)$ respectively. The best way to become familiar with pushforwards and pullbacks is to find these morphisms on your own and use them to establish this bijection.

7.6 BONUS: COSHEAVES

Sheaves come with a cohomology theory because of the directions of their restriction maps, which point from low-dimensional simplices to high-dimensional ones. In order to produce an equal and opposite homology theory, one requires maps going in the other direction; this is achieved by reversing the partial order on the simplices of the base space L.

DEFINITION 7.20. A **cosheaf** over L is a functor $\mathscr{C} : (L, \geq) \to \mathbf{Vect}_{\mathbb{F}}$.

Thus, $\mathscr{C}$ assigns an $\mathbb{F}$-vector space $\mathscr{C}(\tau)$ (called the *costalk*) to each simplex τ of L; and it assigns a a linear map $\mathscr{S}(\tau \geq \tau') : \mathscr{S}(\tau) \to \mathscr{S}(\tau')$ (called the *extension map*) to each coface relation $\tau \geq \tau'$ in L. Moreover, we require the expected axioms to hold:

(1) the map $\mathscr{C}(\tau \geq \tau)$ is the identity on $\mathscr{C}(\tau)$, and
(2) the equality $\mathscr{C}(\tau' \geq \tau'') \circ \mathscr{C}(\tau \geq \tau') = \mathscr{C}(\tau \geq \tau'')$ holds for every triple of simplices $\tau \geq \tau' \geq \tau''$.

All of the constructions and results which have been described for sheaves in this Chapter also admit cosheafy analogues — for instance, every cosheaf $\mathscr{C}$ on L induces a chain complex

$$\cdots \xrightarrow{\partial^{\mathscr{C}}_{k+1}} \mathbf{C}_k(L;\mathscr{C}) \xrightarrow{\partial^{\mathscr{C}}_{k}} \mathbf{C}_{k-1}(L;\mathscr{C}) \xrightarrow{\partial^{\mathscr{C}}_{k-1}} \cdots \xrightarrow{\partial^{\mathscr{C}}_{2}} \mathbf{C}_1(L;\mathscr{C}) \xrightarrow{\partial^{\mathscr{C}}_{1}} \mathbf{C}_0(L;\mathscr{C}) \longrightarrow 0$$

which gives rise to the **homology of** L **with coefficients in** $\mathscr{C}$. Similarly, there are dual notions of étale spaces, pushforwards and pullbacks for cosheaves.

EXERCISES

EXERCISE 7.1. Given two monotone functions $f, f' : K \to \mathbb{R}$ on a simplicial complex K, assume there exists some $\epsilon > 0$ so that $|f(\sigma) - f'(\sigma)| < \epsilon$ holds for every simplex σ of K. Letting $\mathbf{F}_\bullet$ and $\mathbf{F}'_\bullet$ denote the sublevelset filtrations of K with respect to f and f' respectively, show that the barcodes of $\mathbf{H}_k(\mathbf{F}_\bullet K)$ and $\mathbf{H}_k(\mathbf{F}'_\bullet K)$ have bottleneck distance at most ϵ for every $k \geq 0$. [Hint: find an ϵ interleaving of the two persistence modules and use Theorem 6.19]

EXERCISE 7.2. Describe the stalks and restriction maps of the fiber homology sheaves $\mathscr{F}^k_f$ for $k \geq 0$ when f is the inclusion $\partial\Delta(k) \hookrightarrow \Delta(k)$.

EXERCISE 7.3. Let L be a simplicial complex and τ a simplex in L of dimension $k \geq 0$. What are the cohomology groups of L with coefficients in the skyscraper sheaf $\underline{\mathrm{Sk}}_\tau$?

EXERCISE 7.4. Let $f : \partial\Delta(2) \hookrightarrow \Delta(2)$ be the inclusion map and $\mathscr{F}^k_f$ the associated fiber homology sheaf for each $k \geq 0$. Compute the cohomology groups $\mathbf{H}^i(\Delta(2), \mathscr{F}^j_f)$ for all four pairs $0 \leq i, j \leq 1$.

EXERCISE 7.5. Find a sheaf $\mathscr{S}$ on a contractible simplicial complex L for which $\mathbf{H}^1(L;\mathscr{S})$ is nonzero.

EXERCISE 7.6. Show how Corollary 7.13 follows from the argument which was used to prove Proposition 7.12.

EXERCISE 7.7. Show that every morphism $\Phi : \mathscr{S} \to \mathscr{S}'$ of sheaves over a simplicial complex L induces well-defined linear maps $\mathbf{H}^k(L;\mathscr{S}) \to \mathbf{H}^k(L;\mathscr{S}')$ of sheaf cohomology groups.

EXERCISE 7.8. Show that the pullback $f^*\mathscr{S}$ of a sheaf over L across a simplicial map $f : K \to L$ is a sheaf over K

EXERCISE 7.9. Complete the proof of Theorem 7.14.

EXERCISE 7.10. Show that for every simplicial map $f : K \to L$ and each dimension $k \geq 0$, the assignment of fiberwise *cohomology* groups $\tau \mapsto \mathbf{H}^k(\tau/f)$ constitutes a cosheaf over L.